A Variation of the Method Using the Simulation of a Diffusion Process to Characterize the Shapes of Plane Figures*

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Abstract

This is a variation of a previously presented method [1] for characterizing the shapes of plane figures. In addition to retaining the advantages of the original method, this variant includes one more: It is no longer necessary to halt a (simulated) diffusion process during the transient stage; that is, before arriving at an equilibrium. On the contrary, the longer the process takes, the more noticeable the difference becomes between the concave parts and the convex parts of the contours of the figures analyzed.

Key words: shape characterization, contours of plane figures, diffusion process, concavities, convexities.
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1 Introduction

A method was previously described in which the simulation of a diffusion process is used for the characterization of the shapes of plane figures [1]. This method has been used for different purposes by other authors ([2] and [3]). In brief, this method can be described as follows:

1. A digital representation of the figure to be studied must be obtained. Thus the figure will be represented by a connected set of pixels (to which each is assigned a 1 (one)), immersed in an environment made up of pixels in which the figure under consideration does not appear (each one of which is assigned a 0 (zero)). (See **Figure 1**).

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	**1**	**1**	**1**	**1**	0	0	0	0
0	0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0
0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0	0
0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0
0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0
0	0	0	0	**1**	**1**	**1**	**1**	**1**	**1**	**1**	**1**	0	0	0	0
0	0	0	0	0	0	0	0	**1**	**1**	**1**	**1**	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0

Figura 1

The different criteria used to obtain digitalized representations of the figures to be studied will not be discussed in this article.

2. It is supposed that in each one of the pixels of the contour of the digital version of the figure to be studied–which will be referred to simply as the "figure"–initially contains ($t = 0$), a certain number of particles such as 10000 particles.

3. As of $t = 0$, a "simulated" diffusion process takes place with those particles. Thus the pixels on the inside of the figure–which initially lacked particles–will contain an increasing number of particles.

4. The "simulated" diffusion process is halted before the figure arrives at a situation of equilibrium–or uniformity–in regard to the content of particles of each one of the

pixels that make it up. (Note that if the diffusion process were allowed to continue long enough, it would inevitably reach that state of equilibrium in which each one of the pixels of the figure would have the same number of particles.) At what instant is the diffusion process detained? In the paper in which this method was introduced, it was decided to use the instant in which a certain variable (d), whose nature will be specified below, reaches its maximum value. The variable d is the difference between two numbers: the number of particles contained in the pixel (or in each one of the pixels) of the contour containing the greatest number of particles and the number of particles contained in the pixel (or in each one of the pixels) of the contour containing the least number of particles.

5. Once the simulated diffusion has been detained, the following is done:

   (a) Number the pixels of the contour of the figure using one of the two possible directions: clockwise or counterclockwise. The number 1 may be assigned to the pixel of the contour containing the largest number of particles.[1]

   (b) Graph the number of particles contained in each pixel of the contour–for the instant in which the simulated diffusion process was detained–according to the number of the pixel of the contour.

   The regions with the highest numerical value in the resulting graph correspond to the parts of the contour which can be classified as concavities from the perspective of an observer situated inside the figure–whereas the regions with the lowest value in the resulting graph correspond to the parts of the contour which may be classified as convexities from the same perspective. Expressed intuitively, the reason for this result is that there are "difficulties" –or relatively "few possible trajectories"–for the particles to diffuse toward the inside of the figure from the concave parts (from the same perspective) of the contour of the figure. On the contrary, there is a lower degree of difficulty–that is, a larger number of available trajectories–for the particles to diffuse toward the inside of the figure from the convex parts (again from the same perspective) of the contour of the same figure.

It can be noticed that at a given time in the stage before equilibrium–that is, during the "transitory" period–this approach requires stopping the diffusion process. As indicated above, if this process is allowed to continue, all of the pixels of the figure analyzed–even those of the contour– would eventually end up with the same number of particles. The objective of this paper is to present a variation of this approach, such that the differences existing between the contents of the particles of the different pixels of the contour will not cancel each other out if the simulation process is prolonged as long as is desired. On the contrary, if the variation to be described below of the previous approach is used, the

[1]When each of two or more pixels of the contour contains a number of pixels equal to the "largest number"–or maximum–to which reference was made, an algorithm was designed to make it possible to: a) determine to which of these pixels the number 1 should be assigned, or b) conclude that it makes no difference which pixel is given the number 1. (This algorithm will not be specified in this article.)

differences between the contents of the particles of the diverse pixels of the contour will become progressively more significant with regard to the characterization of its different parts, such as concavities and convexities.

2 Description of a variation of the method which uses the simulation of a diffusion process for the characterization of the shapes of plane figures

In the variation of the method considered, the same number of particles is placed in each one of the pixels inside the figure under study. It is supposed, however, that at the initial instant–that is, at $t = 0$–there are no particles contained in the pixels of the contour of the figure. These pixels–those of the contour of the figure–during the simulated diffusion process play the role of "sinks": the particles that diffuse toward them from the pixels inside the figure will be able to enter the pixels in the contour of the figure but will not be able to leave them.

The representation below of any pixel inside a figure is such that its eight neighboring pixels also belong to the inside of the figure–that is, they are not pixels belonging to the contour of that figure (See **Figure 2**).

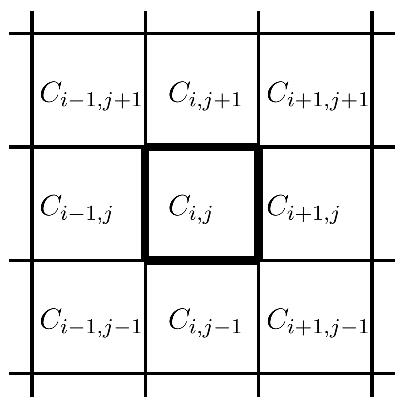

Figura 2

The equation used for the diffusion of particles for that pixel is:

$$N_{i,j}(t+1) = N_{i,j}(t) - 6k\, N_{i,j}(t) + k\,[\, N_{i-1,j}(t) + N_{i,j+1}(t) + N_{i+1,j}(t) + N_{i,j-1}(t)\,] + \frac{1}{2}k\,[\, N_{i-1,j-1}(t) + N_{i-1,j+1}(t) + N_{i+1,j+1}(t) + N_{i+1,j-1}(t)\,]$$

The justification of the above equation is as follows: The time is considered to be composed of equal elemental lapses. The elemental lapse Δt is taken as the unit of time, so that the instant $t+\Delta t$ may be referred to as instant $t+1$. The left-hand member ($N_{i,j}(t+1)$) thus represents the number of particles contained in compartment $C_{i,j}$ at instant $t+1$. This number was made equal to the number of particles contained in $C_{i,j}$ at instant t–$N_{i,j}(t)$– minus the number of particles that were diffused, during the elemental lapse between instants t y $t+1$, from compartment $C_{i,j}$ toward the eight neighboring compartments–$6k\ N_{i,j}(t)$–plus the number of particles that entered during that lapse of time $C_{i,j}$ from $C_{i-1,j}$, $C_{i,j+1}$, $C_{i+1,j}$ and $C_{i,j-1}$ $-k[\ N_{i-1,j}(t)+N_{i,j+1}(t)+N_{i+1,j}(t)+N_{i,j-1}(t)\ ]$– plus the number of particles that entered $C_{i,j}$, during that same lapse, from compartments $C_{i-1,j-1}$, $C_{i-1,j+1}$, $C_{i+1,j+1}$ and $C_{i+1,j-1}$ $-\frac{1}{2}k[\ N_{i-1,j-1}(t)+N_{i-1,j+1}(t)+N_{i+1,j+1}(t)+N_{i+1,j-1}(t)\ ]$–.

It can immediately be seen that two different diffusion constants were used: k–as a diffusion constant between $C_{i,j}$ and the neighboring compartments which have a side in common with it– and $\frac{1}{2}k$ as a diffusion constant between $C_{i,j}$ and the neighboring compartments which have only one vertex in common with it. At first sight, this last diffusion constant seems difficult to justify. How can a diffusion process be conceived as taking place along a vertex, that is, a point, in geometric terminology? It may be thought that, if one wants to find a physical explanation, on the inside of the compartments represented in **Figure 2** there are two little hollow spheres and that actually the diffusion processes–leading to the exchange of particles–takes place between them by means of small tubes connecting any sphere to those located in neighboring compartments. The diameter of some of these small tubes, for example, those connecting the sphere contained in $C_{i,j}$ with those contained in $C_{i-1,j}$, $C_{i,j+1}$, $C_{i+1,j}$ and $C_{i,j-1}$–is greater than the diameter corresponding to other tubes–for example, those that connect the sphere contained in $C_{i,j}$ with those contained in $C_{i-1,j-1}$, $C_{i-1,j+1}$, $C_{i+1,j+1}$ and $C_{i+1,j-1}$. It also can be supposed that the relation between the diameters mentioned was chosen precisely so that the previously mentioned diffusion constants would be fixed at k and $\frac{1}{2}k$.

What is the objective of introducing a diffusion process between "diagonally" placed neighboring compartments? To achieve a certain degree of smotthness in the transitions between the numbers of particles located in these compartments of the contour of the figures analyzed. It should be kept in mind that since these last compartments behave like "drains," there is no diffusion process taking place between them to prevent abrupt changes from occurring between the numbers of particles which were just mentioned. In any case, these abrupt changes are due to the staircase effect of the structure which they adopt, in the digitalized versions of the figures, with some parts contour which in the original versions of these figures are simple, for example, straight segments. The procedure adopted here to "smooth" the transitions mentioned above, or to prevent these abrupt changes considerably consists of: 1) using, as indicated above, two diffusion constants (k and $\frac{1}{2}k$) and 2) averaging the values corresponding to the number of particles which are contained, at the end of the diffusion process, in a certain amout of consecutive pixels of the contour. Part 2 of this procedure will be described below in greater detail.

Following are several examples using the new approach described here for the characterization of shapes of plane figures.

Example 1 *Let the rectangle be that represented in* ***Figure 3.***

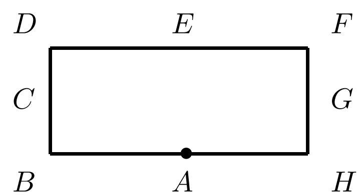

Figura 3

The “point” indicated in the contour of the rectangle represented in **Figure 3** corresponds to the pixel of the contour to which the number 1 was assigned. (Likewise, mention has been made regarding the pixels to which that number was assigned in the rest of the figures which were used as illustrations in this paper, as examples of the application of the variation introduced here.) It can also be seen that the point indicated was assigned the letter A. In addition, there are other letters– B, C, D, E, F, G and H–near the contour of the rectangle. These letters were used to identify certain points of the regions of the rectangle having the highest degrees of concavity or convexity, from the point of view specified above.

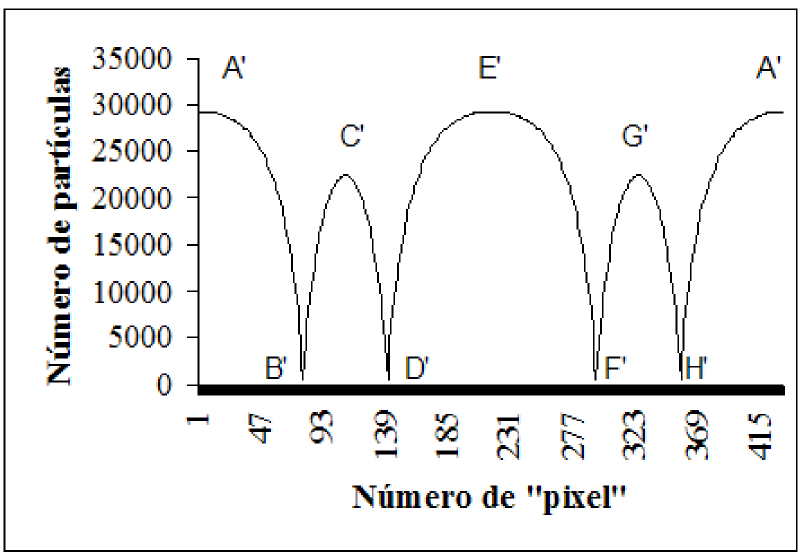

Figure 4

Here, 1000 particles were initially placed in each one of the pixels fo the inside of this rectangle. The diffusion constant k was taken as equal to 0.1–or $k = 0.1$. The simulated diffusion process was carried out until the moment when each one of the pixels of the inside of the rectangle was found to contain a number of particles equal to or less than

10 –or $N_f \leq 10$. + The graph corresponding to the representation of the number of particles in each one of the pixels of the contour according to the numbers assigned to those pixels is presented in **Figure 4**. Using the symbols A', B', C', D', E', F', G' and H', the points of the curve resulting from the application of the procedure described that correspond to the points A, B, C, D, E, F, G and H, respectively, have been specified in this graph.
Note that in this example it was not necessary to find the averages mentioned above due to the fact that given the horizontal position of this particular rectangle, no ladder-like structures appear for the pixels of the contour in the digitalized version of the figure. The situation can be different, as seen in the following example:

Example 2 *Let the rectangle be that represented in* ***Figure 5***

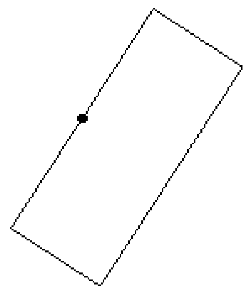

Figure 5

The resulting curve, when applying the procedure described here–except for the final process for obtaining the averages for the pixels of the contour–is represented in **Figure 6**.

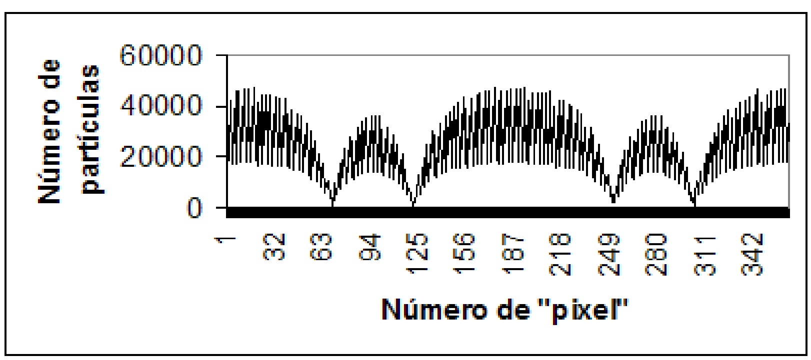

Figure 6

The abrupt "jumps" are outstanding in the number of particles that the pixels of the contour end up containing, once the simulation of the diffusion process is detained. These

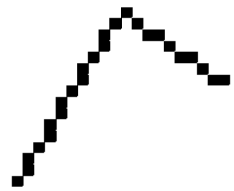

Figure 7

“jumps” are due to the ladder-like structure mentioned above which the sides of the rectangle have in this case. (This effect is caused, of course, by the oblique position of these sides with respect to the retícula of reference pixels.) Part of this ladder-like structure is shown in **Figure 7**.

If one continues to obtain the averages of the last numbers of particles contained in the pixels of each set of 31 consecutive pixels of the contour of the rectangle and checks that those averages as ordinates corresponding to the abscissas of the central pixels of these sets, **Figure 8** is obtained.

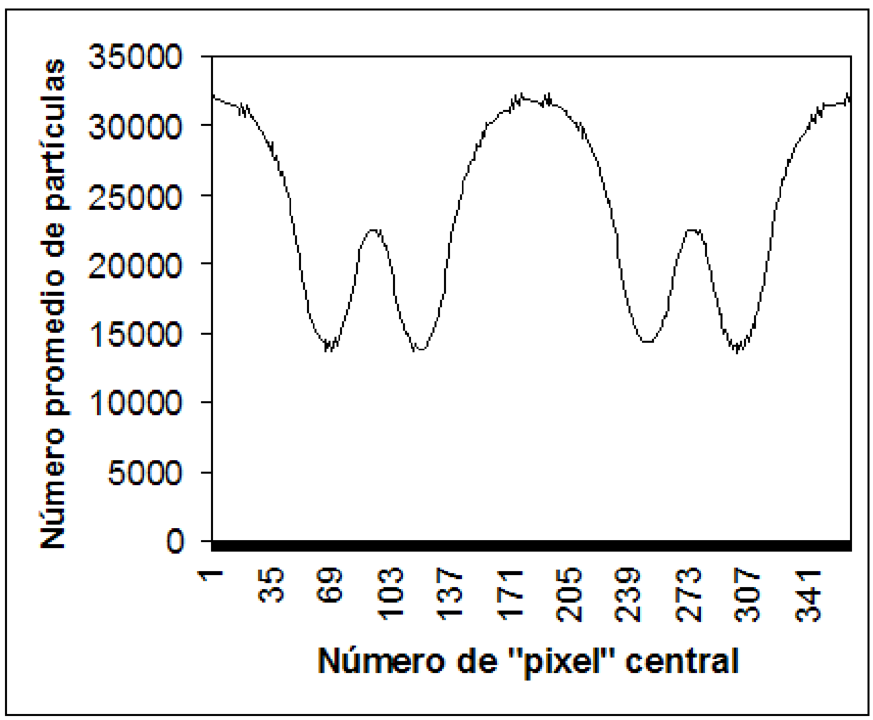

Figure 8

Greater detail is provided below as to which of the averages have been found:

1. Before finding these averages, the values were found for the instant in which the simulation of the diffusion process was detained. That is, the following values are

for the numbers of particles contained in each one of the pixels of the contour, those found between pixel 1 and pixel 31, inclusive.[2]

No. of pixel	No. of particles	No. of pixel	No. of particles	No. of pixel	No. of particles
1	47651	11	32599	21	16872
2	19109	12	45809	22	31116
3	42088	13	17953	23	44006
4	17230	14	32694	24	17269
5	31847	15	45983	25	31422
6	45204	16	18045	26	44028
7	17779	17	32958	27	17219
8	32436	18	46961	28	31257
9	45648	19	18795	29	43684
510	17900	20	41289	30	17063
				31	30938

The average of the numbers of articles according to the data from the above table is equal to 31447. This number corresponds as an ordinate of the abscissa 16, corresponding to the central pixel–16–of the set of pixels: $1, 2, 3, \ldots, 31$.

2. Before finding these averages, the values were found for the instant in which the simulation of the diffusion process was detained. That is, the following values are for the numbers of particles contained in each one of the pixels of the contour, those found between pixel 2 and pixel 32, inclusive.

No. of pixel	No. of particles	No. of pixel	No. of particles	No. of pixel	No. of particles
2	19109	12	45809	22	31116
3	42088	13	17953	23	44006
4	17230	14	32694	24	17269
5	31847	15	45983	25	31422
6	45204	16	18045	26	44028
7	17779	17	32958	27	17219
8	32436	18	46961	28	31257
9	45648	19	18795	29	43684
10	17900	20	41289	30	17063
11	32599	21	16872	31	30938
				32	43174

The average of the numbers of particles according to the data from the above table is equal to 31302. This number corresponds as an ordinate of the abscissa 17, corresponding to the central pixel–17–of the set of pixels: $2, 3, 4, \ldots, 32$, and so forth.

Once the averages obtained are assigned to the different pixels, it is possible that the largest of the averages will not correspond to pixel 1. Thus the pixels should be renumbered taking

[2] Ver el apéndice de este artículo.

into account the values of the averages which have been assigned to them: number 1 is assigned to the pixel to which the largest of the pixels is assigned. If a number equal to the "greatest of the averages" or the maximum average is assigned to each one of two or more pixels of the contour, to which the number 1 will be assigned will be found by using the same algorithm mentioned above in a footnote. Beginning with the pixel to which number 1 is assigned, the remaining pixels are given numbers $2, 3, \ldots, N$, respectively, going clockwise around the contour of the figure.

Example 3 *Let the figure be that represented in **Figure 9.***

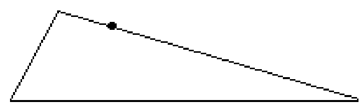

Figure 9

The curve obtained using the procedure described is represented in **Figure 10**.

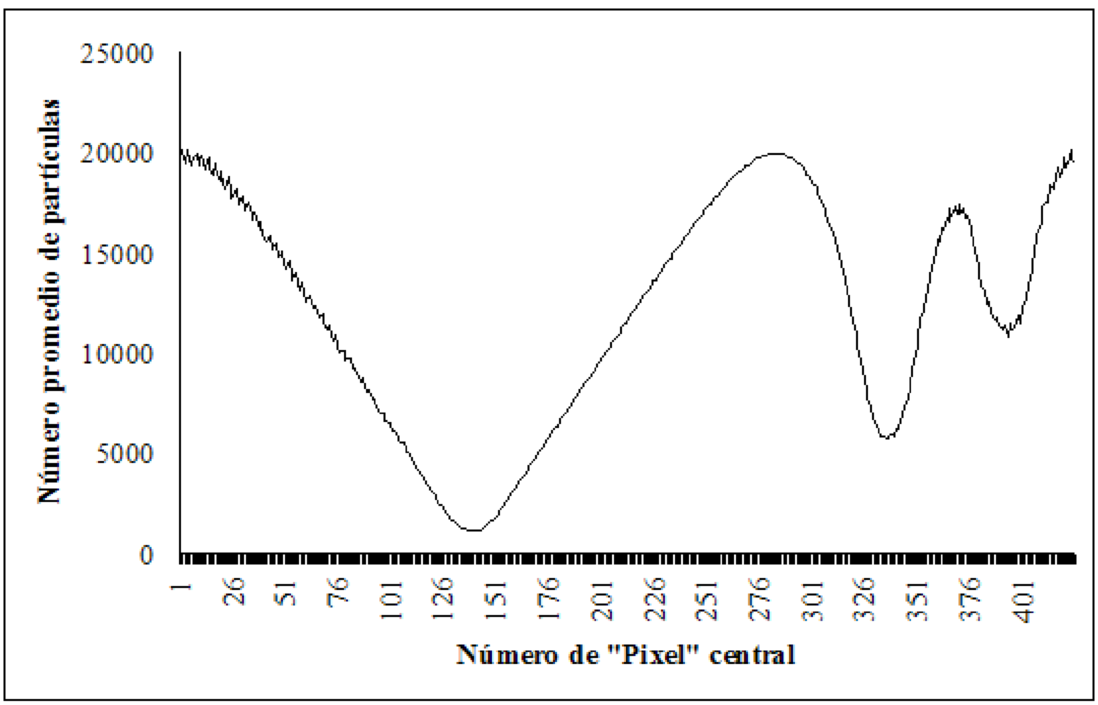

Figure 10

Example 4 *Let the figure be that represented in **Figure 11.***

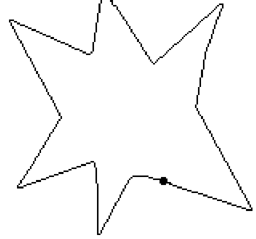

Figure 11

The curve obtained using the procedure described–averaging every 43 pixels–is represented in **Figure 12**.

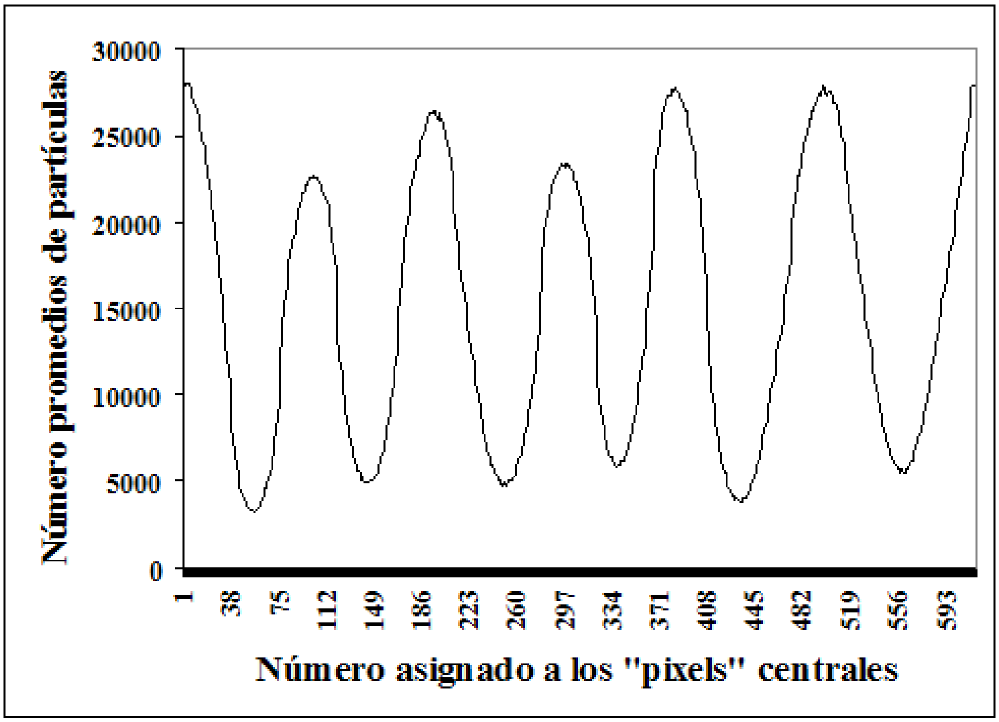

Figure 12

Example 5 *Let the figure be that represented in **Figure 13.***

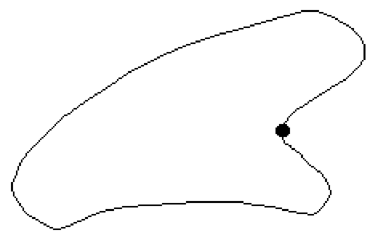

Figure 13

The curve obtained using the procedure described–averaging every 61 pixels–is represented in **Figure 14**.

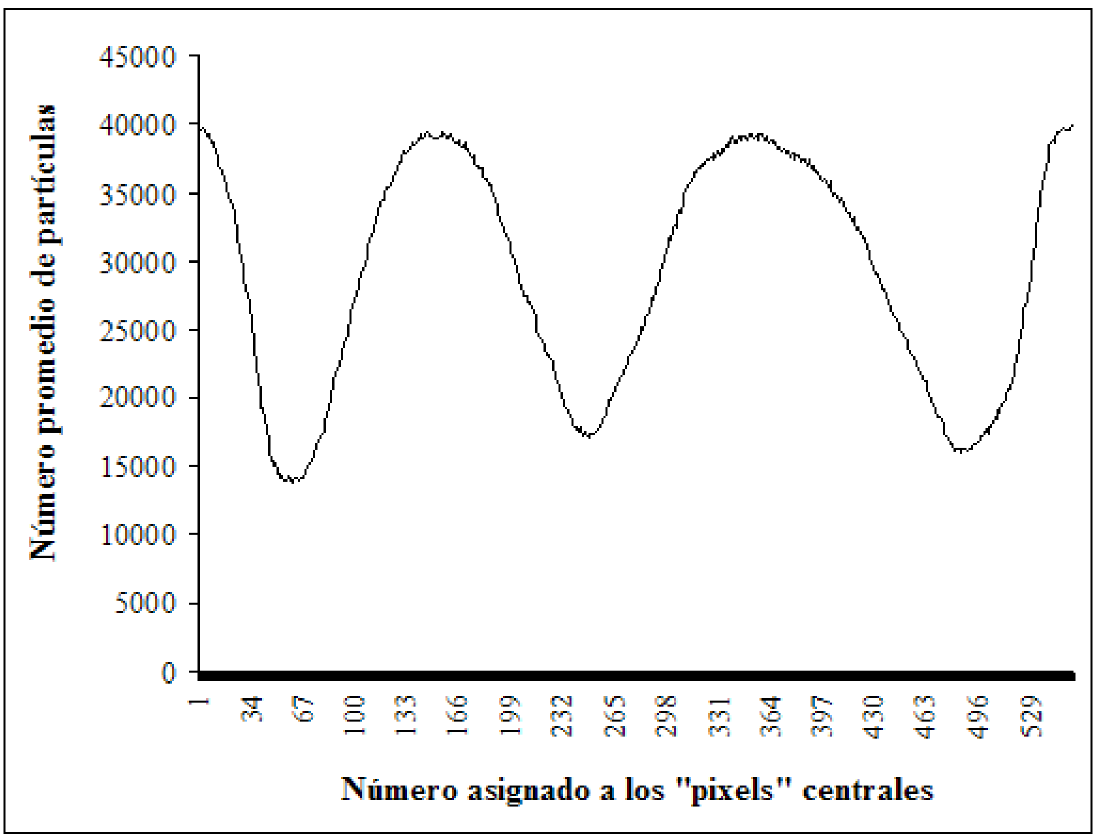

Figure 14

Example 6 *Let the figure be that represented in **Figure 15.***

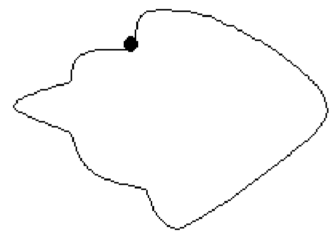

Figure 15

The curve obtained by using the procedure described–averaging every 17 pixels–is represented in **Figure 16**.

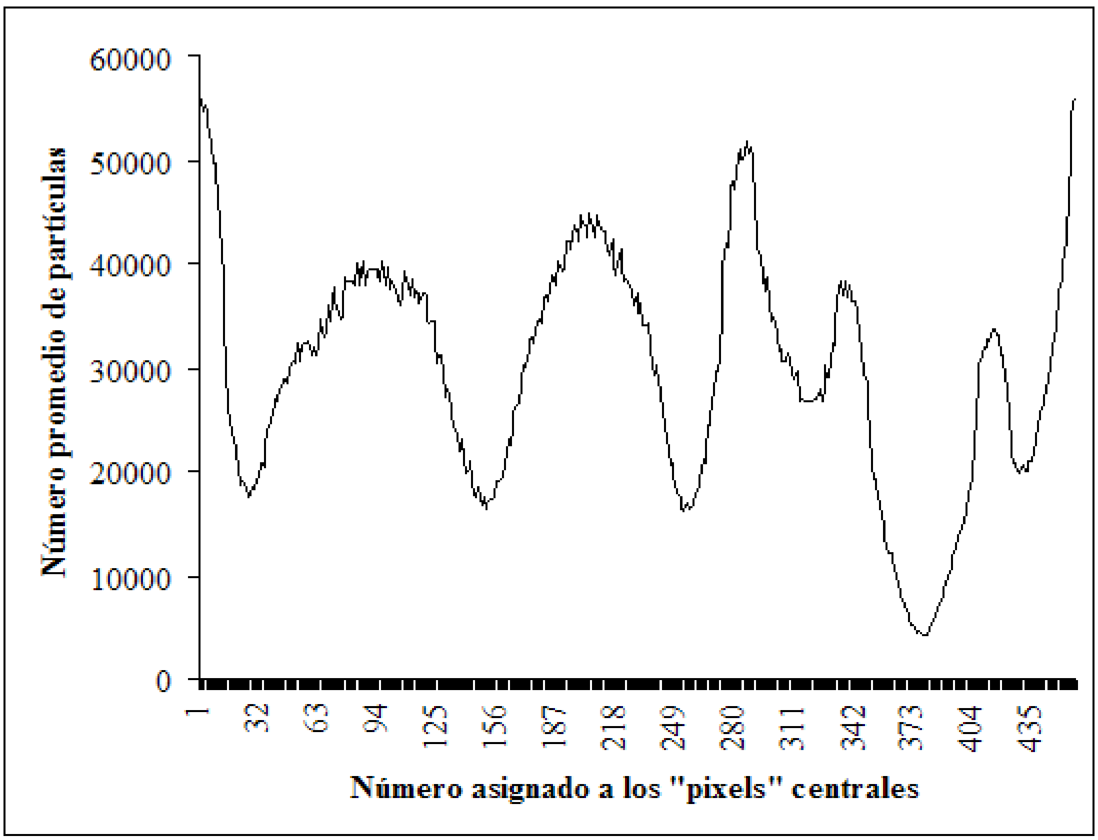

Figure 16

3 Discussion and perspectives

In addition to the advantages of the original method, the variation presented here of the previously described method for the characterization of plane figures has one more: the longer the simulated diffusion process is, the more noticeable the difference is between concave and convex regions of the contours of the figures analyzed.

Let us consider once again, for example, the "star" represented in **Figure 11**. In **Figure 17** one can observe the curves obtained by means of the application of this variation when allowing the diffusion process to last different periods of time, on three different occasions. The initial number of particles in each one of the inside compartments of the figure considered was assumed to be equal to 1000–$N_0 = 1000$. The three different curves graphed in **Figure 17** were obtained by halting the diffusion process when the number of particles in each one of the compartments inside the "star" was: **1)** less than or equal to 700–$N_f \leq 700$; **2)** less than or equal to 300–$N_f \leq 300$; and **3)** less than or equal to 10–$N_f \leq 10$. These three cases correspond respectively to three times: t_1, t_2 and t_3, such that $t_1 < t_2 < t_3$. It may be noted how the general characteristics of the curves obtained are preserved, but with differences between "peaks" and "valleys" (corresponding to convexities and concavities, respectively) progressively greater as the time assigned to the simulated diffusion process increases.

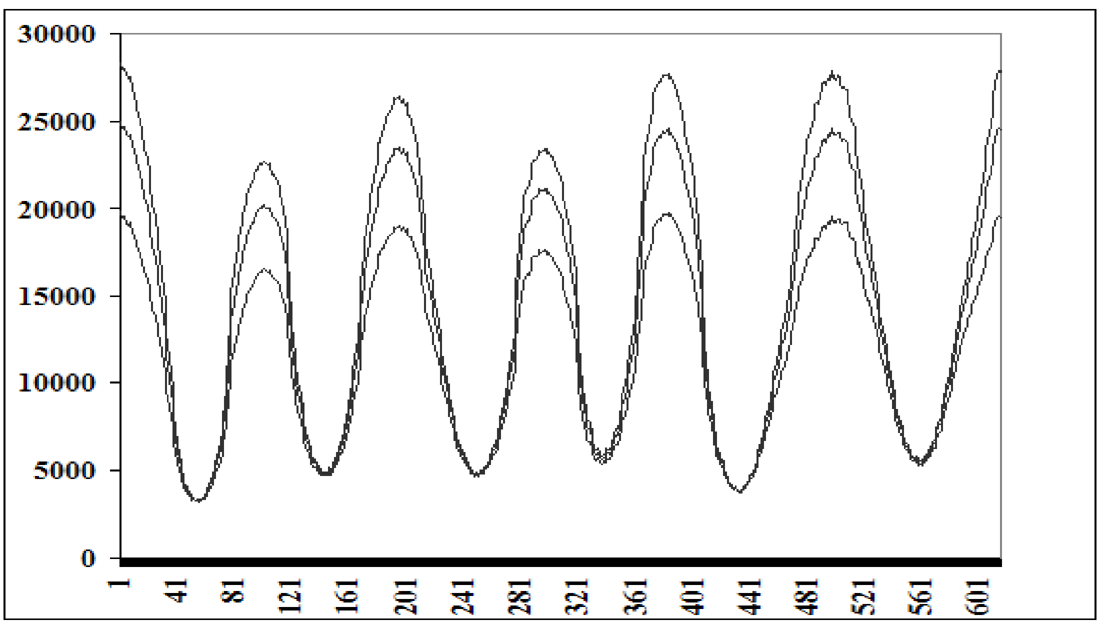

Figure 17

Someone who has not read [1] might ask the following question: "What is the advantage–for the purposes of characterizing the shape of a plane figure–of replacing a closed curve (such as that corresponding precisely to the contour of the figure studied) with an open

curve (like that resulting from the application of the figure considered in the approach described in this article)?" The answer is this: "The second curve is, in principle, independent of the position of the figure analyzed with respect to any reference system."
It is important to note that this article is part of a research program in the field of pattern recognition whose objectives include:

I) finding descriptive mathematical entities for the shapes of plane figures–both simply and multiply connected–which are independent of: **1)** the position of these shapes with respect to any reference system whatsoever, and **2)** the size of the figures in question; and

II) using the results obtained in I) to develop a method for the automatic classification of plane figures according to their shapes.

Appendix
(Related to footnote 2)

Why was "31" chosen as the number of pixels of each one of the sets of consecutive pixels–of the contour of the figure considered–to which reference is made? The answer is as follows:

a) The number chosen should be odd due to the fact that the average which is computed, for each one of the sets will be represented as the ordinate corresponding to an abscissa which will correspond to the central pixel of each set. The sets of pixels consecutive to those to which an even number of pixels belongs will lack a pixel which can be considered as the "central" one, whereas for any set of consecutive pixels it is possible, with no ambiguity at all, to consider one of those pixels as the "central" one of the set in question.

b) A certain amount of arbitrariness must be recognized in the choice of an odd number of pixels pertaining to each set of those considered. Two criteria which must be taken into consideration when making this choice are as follows:

   I) The larger the odd number chosen is, more "abrupt jumps," such as those mentioned above, are prevented. But that is achieved at the expense a loss of sensibility in detecting the concave and convex regions of the figures analyzed. (If that number were made equal to the total number of pixels of the contour, the resulting curve for any figure, when applying the method described here, would be a segment of the line parallel to the axis of the abscissas.)

   II) It is suggested–and this is a heuristic rule for which no attempt will be made to justify in this paper–that the number of pixels be approximately equal to half of the number of pixels composing the smallest structure, making up the figure analyzed, which should be adequately represented in the curve obtained using the procedure desribed. In the example in **Figure 3**, the shortest side of the rectangle is made up of approximately 60 pixels. Thus, 31 was chosen as the number of pixels for each one of the sets mentioned. (This topic will be covered from a more technical perspective in another article.)

4 References

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