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A NEW METHOD FOR THE CHARACTERIZATION OF CLUSTERS

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Abstract

This new method for characterizing clusters is based on the simulation of a diffusion-like process. A resolution-parameter—R—is introduced such that when assigned successive values from an increasing sequence, it is possible to detect the following:

(a) a unique cluster which can be visualized as an object with no internal structure;

(b) a set of n first-order subclusters—given they exist—which are constituents of the cluster mentioned in (a);

(c) n sets of second-order subclusters—each of which are constituents of one of the first-order subclusters mentioned in (b)—and so on, successively.

Convexity is not required either for the cluster mentioned in (a) or for the subclusters of different orders. Although in this paper the method presented is applied to bidimensional objects, it may be generalized for the n-dimensional case.

Keywords: characterization of clusters, diffusion process, resolution-parameter.

Resumen

Se presenta un nuevo método —basado en la simulación de un proceso semejante al de la difusión— para la caracterización de "clusters". Se introduce un "parámetro de resolución —R—" tal que al asignársele sucesivos valores de una secuencia creciente de ellos se permite detectar:

(a) un único cluster que es visualizado como un objeto sin estructura interna;

(b) un conjunto de n subclusters de primer orden —si existen— constitutivos del cluster al que se hizo referencia en (a);

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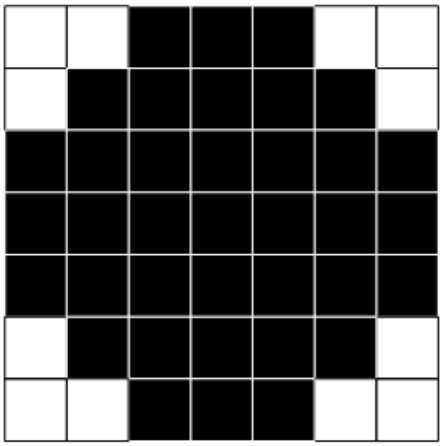


Figure 1: An object composed of 37 pixels which were assigned the color black

(c) n conjuntos de subclusters de segundo orden —los que integran cada uno de los subclusters de primer orden a los que se hizo referencia en (b)—, y así sucesivamente.

La característica de convexidad no es requerida ni para el cluster mencionado en (a) ni para los subclusters de los diversos órdenes. Si bien en este trabajo el método presentado es aplicado a objetos bidimensionales, es factible, en principio, generalizarlo para el caso n-dimensional.

Palabras clave: caracterización de "clusters", proceso de difusión, parámetro de resolución.

Mathematics Subject Classification: 91C20, 60J60.

1 Introduction

The objective of this article is to present a new method for characterizing clusters comprised of bidimensional objects. A review of other approaches for this problem may be found in [1].

It will be supposed that some procedure will be used to "digitalize" the objects to be studied, so that the basic components of these objects will be pixels. In other words, once digitalized, every object of the type considered in this article is comprised of a certain number of pixels. In figure 1 this idea is illustrated using the representation of an object composed of 37 pixels which have been indicated in black. The remaining 12 pixels in figure 1 are not part of the digitalized version of the object under study and have been indicated in white.

This method for the characterization of clusters is applicable to digitalized "versions" of bidimensional objects.

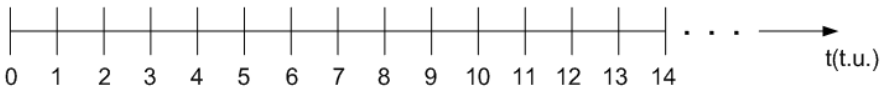


Figure 2: The time axis

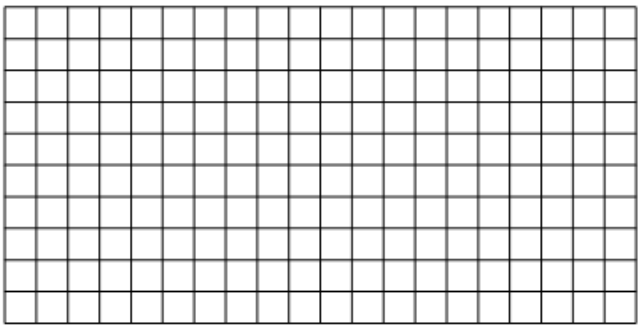


Figure 3: Region of $m \times n$ pixels such that $m = 20$ and $n = 10$

For the purposes of this article "characterization of clusters" will be understood as a way of presenting each cluster under study which will make it possible to identify the different order subclusters comprising it.

2 Description of the Method

This method is based on the supposition that two processes are taking place; they will be described in this section. The state of these processes will be characterized as a function of time. The time axis is shown in figure 2, and the expression "t.u." refers to a time unit. For the purposes of this paper the particular time unit used is not relevant; in other words, it is not necessary to specify, for example, whether it is a second, a millisecond, or some other unit. When using expressions such as $t = 3$ and $t = 19$, it will be understood that reference is being made to $t = 3$ t.u. and $t = 19$ t.u., respectively. No explicit mention will be made below of the unit "t.u."

The region under study is a rectangular area composed of $m \times n$ pixels. (If $m = n$, this area is a square.) For example, a region of 20×10 pixels is shown in figure 3.

Consider any pixel of the region under study. Let i and j be the coordinates—abscissa and ordinate, respectively—corresponding to that pixel, or more precisely, to the center of that pixel, in a system of orthogonal cartesian coordinates, used as a reference system. This pixel will be known as $P_{i,j}$, and will be surrounded by four "adjoining neighbors": $P_{i-1,j}$, $P_{i,j+1}$, $P_{i+1,j}$, and $P_{i,j-1}$. $P_{i,j}$ shares one side with each of these "adjoining neighbors." In

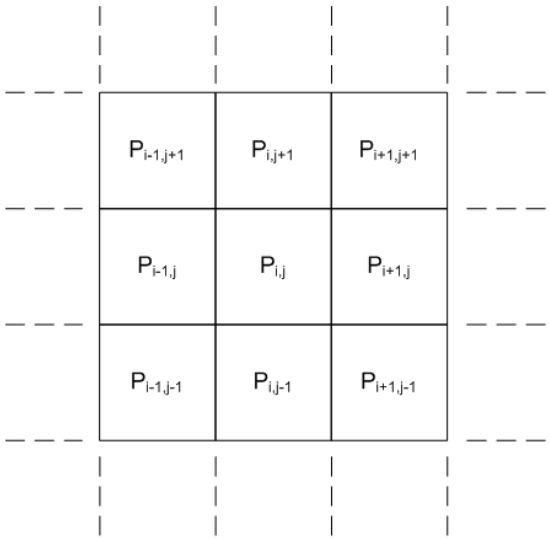


Figure 4: Pixel $P_{i,j}$, its "adjoining neighbors"—$P_{i-1,j}$, $P_{i,j+1}$, $P_{i+1,j}$, and $P_{i,j-1}$.—and its "distant neighbors"—$P_{i-1,j-1}$, $P_{i-1,j+1}$, $P_{i+1,j+1}$, and $P_{i+1,j-1}$

addition, $P_{i,j}$ has in its environment four "distant neighbors": $P_{i-1,j-1}$, $P_{i-1,j+1}$, $P_{i+1,j+1}$, and $P_{i+1,j-1}$. $P_{i,j}$ shares a vertex with each of these "distant neighbors." In figure 4 pixel $P_{i,j}$ can be seen with its "neighbors."

The name "inside pixel" of the region under consideration will be given to every pixel of that region provided that its eight "neighbors" also belong to that region. The name "edge pixel" of the region under consideration will be given to every pixel whenever at least one of its "neighbors" does not belong to that region.

First, let it be accepted that $P_{i,j}$ is an "inside pixel" of the region under study. Suppose as well that in the center of every pixel in that region there is a small hollow sphere inside of which there may be a certain mass of gas. It will also be supposed that these tiny spheres do not exist in pixels not belonging to the region under study.

Consider the tiny sphere corresponding to $P_{i,j}$. This sphere is connected by 8 small tubes to the tiny spheres of the 8 pixels neighboring $P_{i,j}$; that is, $P_{i-1,j}$, $P_{i-1,j+1}$, $P_{i,j+1}$, $P_{i+1,j+1}$, $P_{i+1,j}$, $P_{i+1,j-1}$, $P_{i,j-1}$, and $P_{i-1,j-1}$.

The physical characteristics of the small tubes have been adjusted in such a way that the diffusion coefficient—k—of gas corresponding to the small tubes that connect a tiny sphere of a pixel to the tiny spheres of the pixels which are its "adjoining neighbors" has a value equal to double the diffusion coefficient—$\frac{1}{2}k$—corresponding to the small tubes connecting the tiny sphere of that pixel to those of the pixels that are its "distant neighbors." Thus, for example, the diffusion coefficient of gas k corresponding to the small tubes connecting the sphere contained in $P_{i,j}$ with the tiny spheres contained in $P_{i-1,j}$, $P_{i,j+1}$, $P_{i+1,j}$, and $P_{i,j-1}$ has a value twice that of the diffusion coefficient $\frac{1}{2}k$ corresponding to the small tubes connecting the tiny sphere contained in $P_{i,j}$ with the

tiny spheres contained in $P_{i-1,j-1}$, $P_{i-1,j+1}$, $P_{i+1,j+1}$, and $P_{i+1,j-1}$.

The tiny spheres corresponding to the pixels in which there are objects present will be considered "sources." The number of molecules of gas in these spheres will be considered to be constant, independently of the number of molecules of gas which may be exchanged with the tiny spheres of the neighboring pixels by means of a gas diffusion process. Therefore, if the tiny sphere in $P_{i,j}$ is a "source" sphere (due to the fact that in $P_{i,j}$ there is an object present), then for the number of molecules of gas in the tiny sphere the following holds:

$$N_{i,j}(t) = N_{i,j}(t-1) = \textit{constant}$$

If, on the other hand, the tiny sphere corresponding to $P_{i,j}$ is *not* a "source" sphere (due to the fact that in $P_{i,j}$ there are no objects present), then the number of molecules in that sphere will be determined by two processes: a) the gas diffusion process mentioned above, and b) a process of "decay"—that is, the elimination of the molecules of gas produced by some mechanism whose nature is not relevant for the purposes of this paper. Thus it will be accepted that if the tiny sphere corresponding to $P_{i,j}$ is not a "source" sphere, then the number of molecules of gas in that sphere can be expressed as follows:

$$\begin{aligned} N_{i,j}(t) \;=\; & N_{i,j}(t-1) - 4kN_{i,j}(t-1) - 2kN_{i,j}(t-1) + k(N_{i-1,j}(t-1) + N_{i,j+1}(t-1) \\ & + N_{i+1,j}(t-1) + N_{i,j-1}(t-1)) + \frac{1}{2}k(N_{i-1,j+1}(t-1) + N_{i+1,j+1}(t-1) \\ & + N_{i+1,j-1}(t-1) + N_{i-1,j-1}(t-1)) - k_{\mathrm{d}}N_{i,j}(t-1) \end{aligned}$$

The expression $-k_{\mathrm{d}}N_{i,j}(t-1)$ is the term of decay mentioned above. The interpretation of the other terms of the righthand member of the preceding equation is obvious if this gas diffusion process is taken into account.

The simulation of a diffusion process has been used previously in the field of pattern recognition for the characterization of the shapes of bidimensional objects [2, 3].

Nevertheless, these considerations about the diffusion and decay processes may be left aside, and a purely heuristic mathematical approach may be adopted with respect to certain numerical values as functions of time, for the different pixels. Let $N_{i,j}$ be the numerical value corresponding to $P_{i,j}$; then the following may be accepted:

$$N_{i,j}(t) = N_{i,j}(t-1) = \textit{constant}, \text{ if and only if in } P_{i,j} \text{ there is an object} \qquad (1)$$

and

$$\begin{aligned} N_{i,j}(t) \;=\; & N_{i,j}(t-1) - 4kN_{i,j}(t-1) - 2kN_{i,j}(t-1) + k(N_{i-1,j}(t-1) \\ & + N_{i,j+1}(t-1) + N_{i+1,j}(t-1) + N_{i,j-1}(t-1)) + \frac{1}{2}k(N_{i-1,j+1}(t-1) \\ & + N_{i+1,j+1}(t-1) + N_{i+1,j-1}(t-1) + N_{i-1,j-1}(t-1)) - k_{\mathrm{d}}N_{i,j}(t-1) \\ & \text{if and only if in } P_{i,j} \text{ there is no presence of an object.} \end{aligned} \qquad (2)$$

If $P_{i,j}$ is not an inside pixel of the region under study (that is, if it is a pixel from the edge of that region), then equation (1) remains valid, but equation (2) must be modified

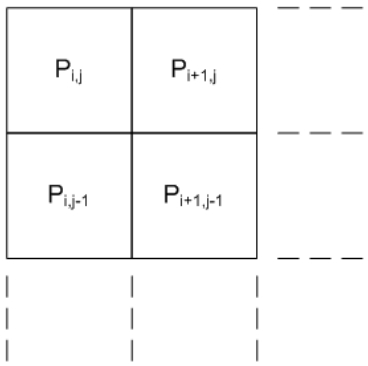


Figure 5: The name $P_{i,j}$ has been assigned to the pixel corresponding to the above lefthand "corner" of the region under study.

appropriately. Suppose, for example, that $P_{i,j}$ is a pixel corresponding to one of the four "cormers" of the region under study. This situation has been illustrated in figure 5.

In the physical processes considered above, since it was supposed that there are no tiny spheres in pixels not belonging to the region considered, it is obvious that the equation corresponding to $P_{i,j}$—when there is no presence of an object in that pixel—is as follows:

$$\begin{aligned} N_{i,j}(t) &= N_{i,j}(t-1) - 2kN_{i,j}(t-1) - \frac{1}{2}kN_{i,j}(t-1) + k(N_{i+1,j}(t-1) \\ &\quad + N_{i,j-1}(t-1)) + \frac{1}{2}kN_{i+1,j-1}(t-1) - k_dN_{i,j}(t-1) \end{aligned}$$

For another example of the treatment corresponding to an "edge pixel," consider now that $P_{i,j}$ is part of the left edge of the region to be considered, as illustrated in figure 6.

In this case if in $P_{i,j}$ there is no presence of an object then for that pixel the following equation will be valid:

$$\begin{aligned} N_{i,j}(t) &= N_{i,j}(t-1) - 3kN_{i,j}(t-1) - kN_{i,j}(t-1) + k(N_{i,j+1}(t-1) + N_{i+1,j}(t-1)) \\ &\quad + N_{i,j-1}(t-1) + \frac{1}{2}k(N_{i+1,j+1}(t-1) + N_{i+1,j-1}(t-1)) - k_dN_{i,j}(t-1) \end{aligned}$$

If one wishes to compute the numerical value corresponding to each pixel—the number of molecules of gas, according to the physical interpretation described—as a function of time, the values of the sixteen parameters used in equations (1) and (2) must be established. These values may be changed—if necessary—for different distributions of the objects under study. Each time the method is applied the values of the following parameters should be provided:

I. The constant numerical value corresponding to each source pixel

II. The initial numerical value corresponding to each pixel which is not a source pixel (In the examples below this value is equal to 0.)

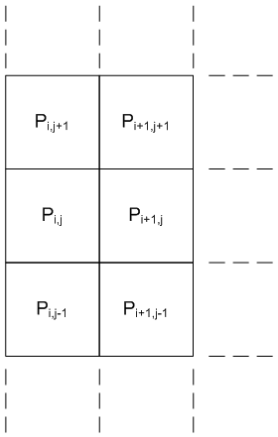


Figure 6: In this case pixel $P_{i,j}$ is part of the lefthand edge of the region considered.

III. The value of the coefficient k—one of the diffusion coefficients, according to the physical interpretation provided. (Beginning with k, the value of the other coefficient of the same type—$\frac{1}{2}k$—is computed.)

IV. The value of the coefficient k_{d}—the "decay coefficient," according to the physical interpretation provided.

Once the numerical values of the parameters mentioned in I, II, III, and IV have been established, equations (1) and (2)—used iteratively—will permit the computation of the values $N_{i,j}$—$i = 1, 2, ..., m$ and $j = 1, 2, ..., n$—corresponding to each pixel of the region considered, for any constant t—$t > 0$.

First of all, the method proposed requires the computation of the numerical values $N_{i,j}$ for each pixel in the region considered, at a certain instant t_f. (How to determine t_f will be specified below.) Second, it is indispensable to introduce the notion of "resolution parameter"—R—and explain how it is used for the characterization of the clusters under study.

For each instant—as of $t = 1$—and each pixel of the region considered, the following expression is computed:

$$|N_{i,j}(t) - N_{i,j}(t-1)|;\ i = 1, 2, \ldots, m; j = 1, 2, \ldots, n$$

The name t_f will be given to the first t for which the following equation is satisfied for all the pixels of the region considered:

$$|N_{i,j}(t) - N_{i,j}(t-1)| < 1$$

In other words, the combination of the process of diffusion with that of particle decay leads the region studied to a "stationary state" in which changes are no longer taking place in the numerical values $N_{i,j}$ corresponding to those pixels. The instant t_f is an instant for

which the region considered arrives at a state very near the above-mentioned "stationary state."

The "resolution parameter" R is simply another parameter to which a numerical value is assigned that is compared systematically with the numerical values $N_{i,j}$ corresponding to the diverse pixels, at the instant t_f. The color black is assigned to each pixel $P_{i,j}$ such that $N_{i,j}(t_f) \geq R$, so that it becomes a constituent of the image that can be detected with the numerical value assigned to R. In addition, the color white is assigned to each pixel $P_{i,j}$ such that $N_{i,j}(t_f) < R$, so that it becomes a constituent of the white background on which the black image mentioned can be seen. If R is assigned a very low value, only one cluster will appear in the region considered; in other words, when observed with very low resolution, it is not possible to distinguish between the objects present. In this situation, all of the objects are "visualized" as a single cluster—the zero-order cluster. As the value of R increases, the number and shape of the clusters present are defined with increasing precision. As shown in the examples below in which this method is applied, first-order clusters may be distinguished for a certain value R. With a higher value of R, second-order clusters may be distinguished, and so on successively. The most detailed image possible of the objects present in the region under study is achieved with the method proposed when the value of R reaches the value assigned to the numerical value of the source pixels.

3 Examples of the application of the above method

The values of the parameters referred to above will be specified for each of the examples of the application of the method presented for the characterization of clusters.

3.1 Example 1

In this case, the region under study is a 800×800-pixel square.

The values of the parameters mentioned above were assigned as follows:

I. For each source pixel $P_{i,j}$, $N_{i,j}(t) = N_{i,j}(t-1) = 8 \times 10^8$;

II. For each pixel $P_{i,j}$ which is not a source pixel, $N_{i,j}(0) = 0$;

III. $k = 0.15$;

IV. $k_{\mathrm{d}} = 0.2$

Given the above values, computer simulation was used to find the following value for t_f:

$$t_f = 71$$

The images shown in figures 7(a), 7(b), 7(c), and 7(d) were obtained when R was assigned the values of 1, 4×10^5, 60×10^6, and 800×10^6 respectively.

In figure 7(a) it can be observed that for $R = 1$, only one cluster may be seen—a zero-order cluster. In figure 7(b) it may be noted that when assigning R the value of 4×10^5, two first-order subclusters are visible. The one on the right has a hole—or in

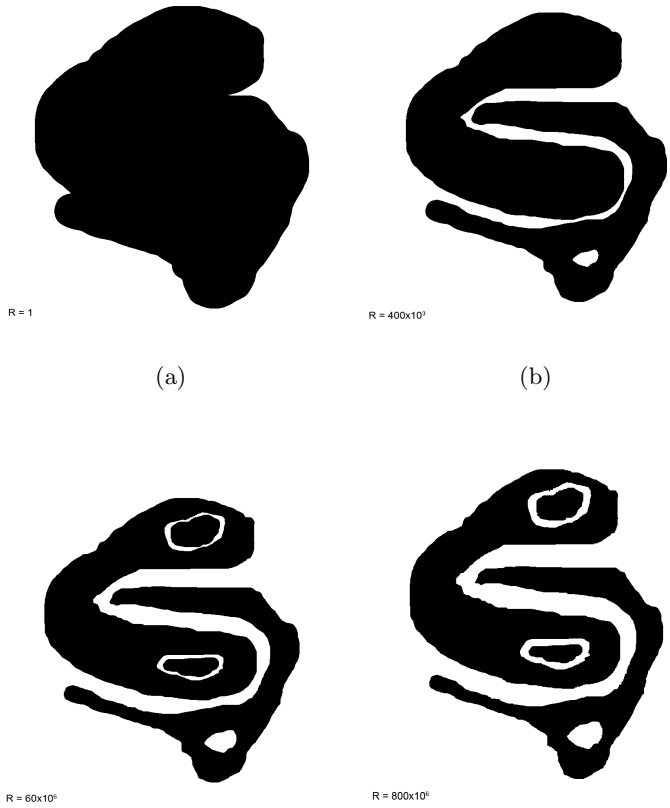


Figure 7: Clusters for Example 1.

topological terms, it has a doubly-connected domain. In figure 7(c) it can be seen that when assigning R the value of 60×10^6, the first-order subcluster situated at the left in figure 7(b) has been transformed into three second-order subclusters. In figure 7(d) it can be noted that when giving R the value of 800×10^6, the subclusters visible in 7(c) have undergone slight changes in shape.

3.2 Example 2

In this case, the region under study is a 800×800-pixel square.

The values of the parameters mentioned above were assigned as follows:

I. For each source pixel $P_{i,j}$, $N_{i,j}(t) = N_{i,j}(t-1) = 8000 \times 10^6$;

II. For each pixel $P_{i,j}$ which is not a source pixel, $N_{i,j}(0) = 0$;

III. $k = 0.15$;

IV. $k_{\mathrm{d}} = 0.2$

Given the above values, computer simulation was used to find the following value for t_f:

$$t_f = 81$$

In this example figures 8(a) and 8(b) are very similar to figures 7(a) and 7(b) respectively. As the value of R is increased, however, a different structure can be observed for the diverse subcusters, in comparison with that found in the first example of the application of this method.

3.3 Example 3

In this case, the region under study is a 800×800-pixel square.

The values of the parameters mentioned above were assigned as follows:

I. For each source pixel $P_{i,j}$, $N_{i,j}(t) = N_{i,j}(t-1) = 80 \times 10^3$;

II. For each pixel $P_{i,j}$ which is not a source pixel, $N_{i,j}(0) = 0$;

III. $k = 0.1$;

IV. $k_{\mathrm{d}} = 0.2$

Given the above values, computer simulation was used to find the following value for t_f:

$$t_f = 33$$

In figure 9(a) it can be observed that for $R = 1$, only one cluster is seen—a zero-order cluster. In figure 9(b) it may be noted that when assigning R the value of 10^3, six first-order subclusters are visible, in such a way that 5 of them are "immersed" in the resulting subcluster. In figure 9(c) it may be noted that when assigning R the value of 10×10^3, each of the five "inside" first-order subclusters is comprised of second-order subclusters. If these five "inside" first-order subclusters are observed going from the top right down counter-clockwise, then it can be noted that the first, second, third, fourth and fifth of these first-order subclusters are made up of 4, 4, 5, 4 and 4 second-order subclusters respectively. In figure 9(d) it can be noted that when assigning R the value of 80×10^3, the subclusters visible in 9(c) have undergone slight changes in shape.

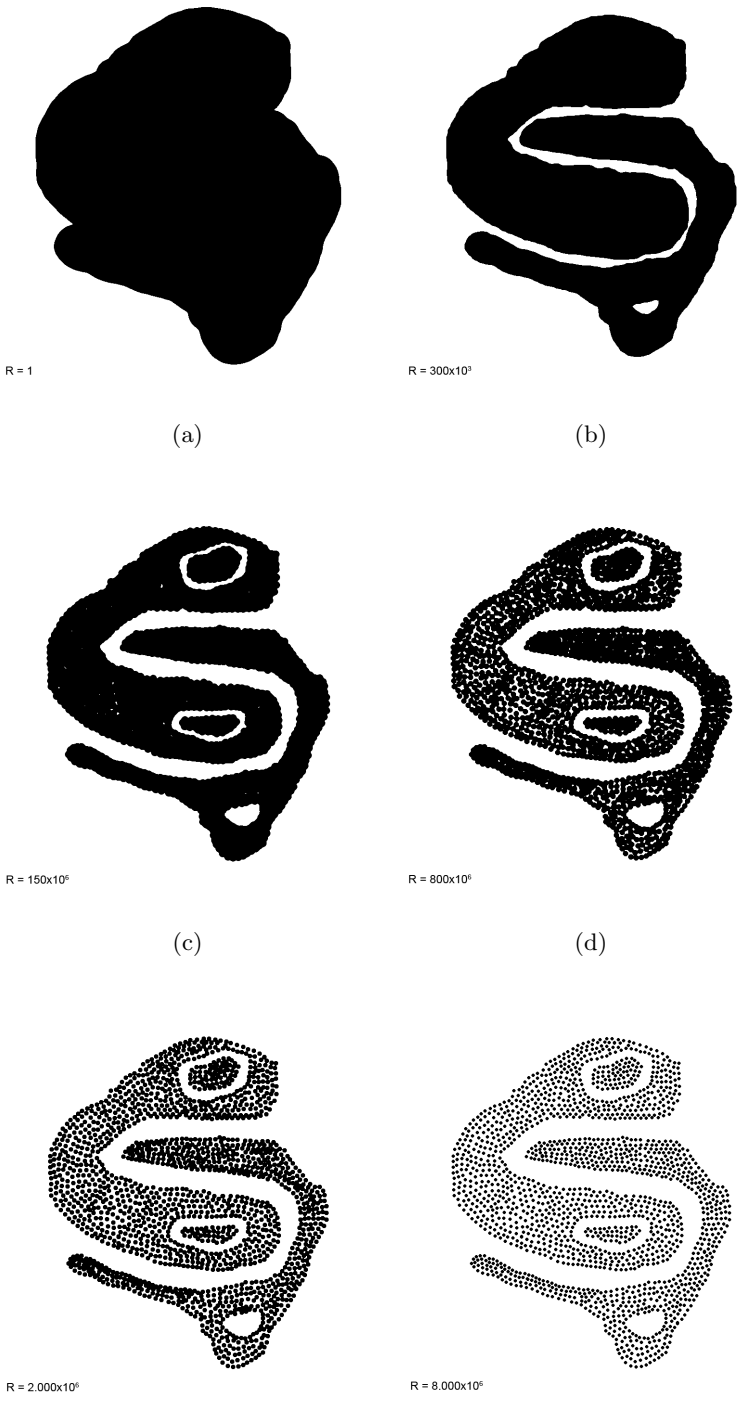
3.4 Example 4

In this case, the region under study is a 250×250-pixel square.

The values of the parameters mentioned above were assigned as follows:

I. For each source pixel $P_{i,j}$, $N_{i,j}(t) = N_{i,j}(t-1) = 800 \times 10^3$;

II. For each pixel $P_{i,j}$ which is not a source pixel, $N_{i,j}(0) = 0$;

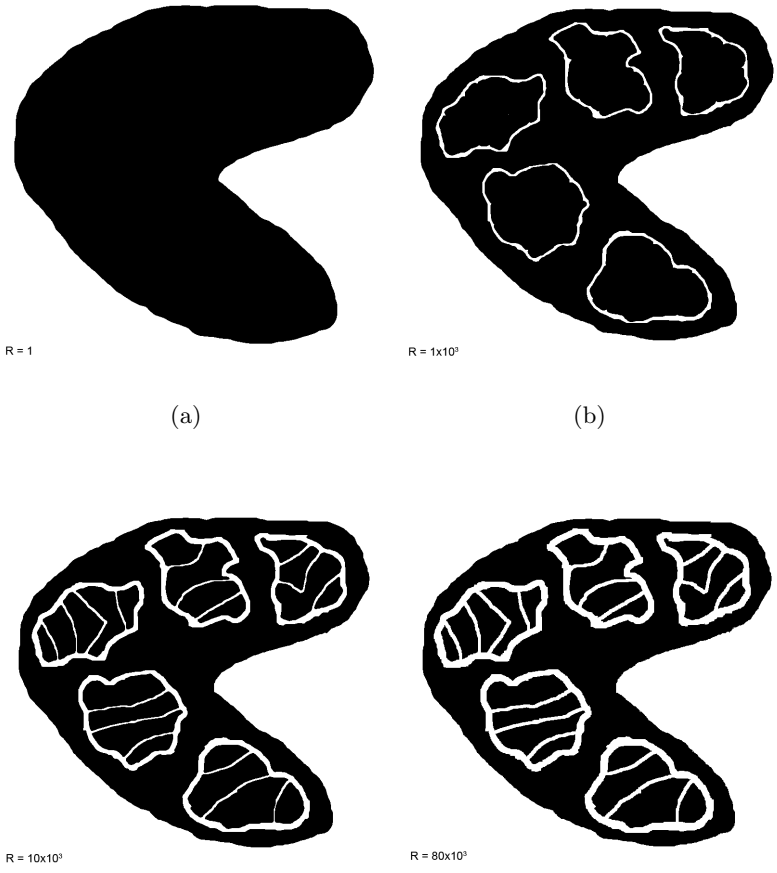


(a) (b)

(c) (d)

(e) (f)

Figure 8: Clusters for Example 2



(a) (b)

(c) (d)

Figure 9: Clusters for Example 3

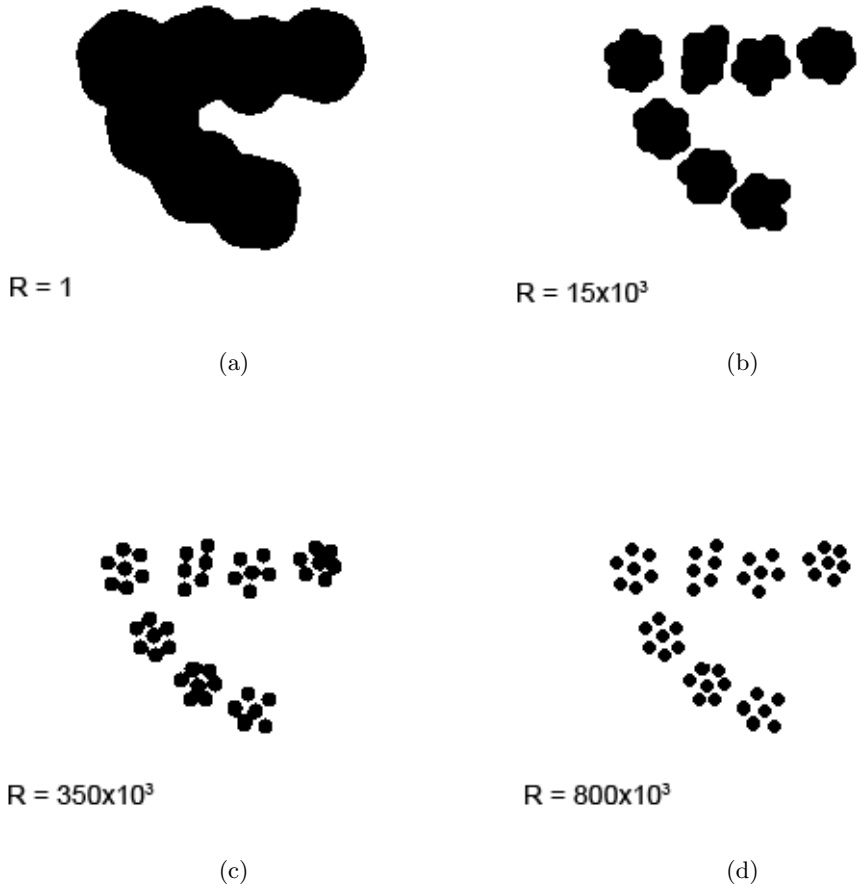


Figure 10: Clusters for Example 4

III. $k = 0.1$;

IV. $k_{\mathrm{d}} = 0.2$

Given the above values, computer simulation was used to find the following value for t_f:

$$t_f = 42$$

Our readers may analyze for themselves the results shown in figures 10(a), 10(b), 10(c) and 10(d).

4 Prospects

One interesting characteristic of the method presented here for the characterization of clusters is that it can be applied without imposing geometrical or topological restrictions on the objects that constitute the clusters under study. The only requirement is that they be comprised of pixels.

The applications of this method were limited in this article to the case of clusters made up of bidimensional objects. The obvious and natural extension of this method to the treatment of clusters of objects of more than two dimensions will be discussed elsewhere.

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